Hmk Q5 / (z_n) sequence

$$z_n = \left(\frac{1}{2+i}\right)^n$$

$f(z) = |z|$ is a continuous fcn.

$$\lim_{z\to 0} g(z) = \lim_{|z|\to 0} g(z)$$

$z \to 0$ means $|z-0| = |z| = ||z|-0|$ is getting small

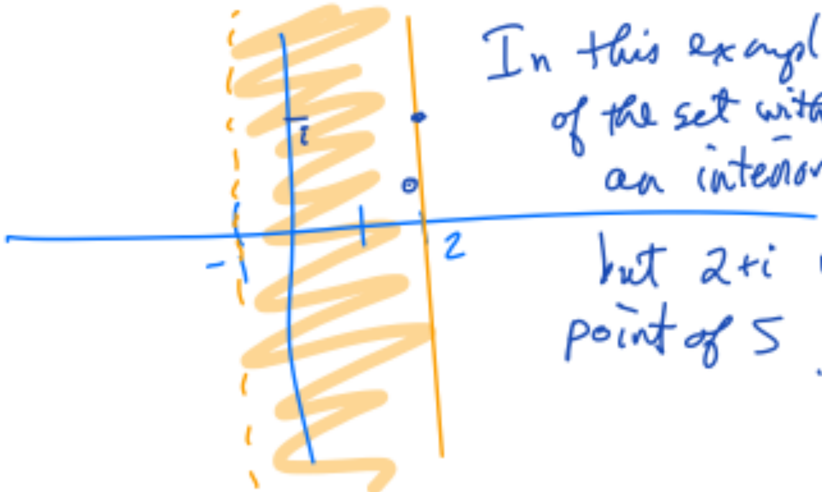
$z \to 3+4i$ means $|z-3-4i|$ is getting small.

Back to <u>Topology & Set Theory</u>

The only two sets in $\mathbb{C}$ that are both open & closed are $\emptyset$ and $\mathbb{C}$.

Given any set $S \subseteq \mathbb{C}$, an <u>interior point</u> of S is a point $w \in S$ s.t. $\exists\, \varepsilon > 0$ s.t. $B(w,\varepsilon) \subseteq S$. [for an open set, <u>every</u> point of the set is an interior point].

e.g. $S = \{ z \in \mathbb{C} \mid -1 < \mathrm{Re}(z) \leq 2 \}$.



In this example, every point of the set with $\mathrm{Re}(z) < 2$ is an interior pt. But e.g. $2+i \in S$, but $2+i$ is not an interior point of S.

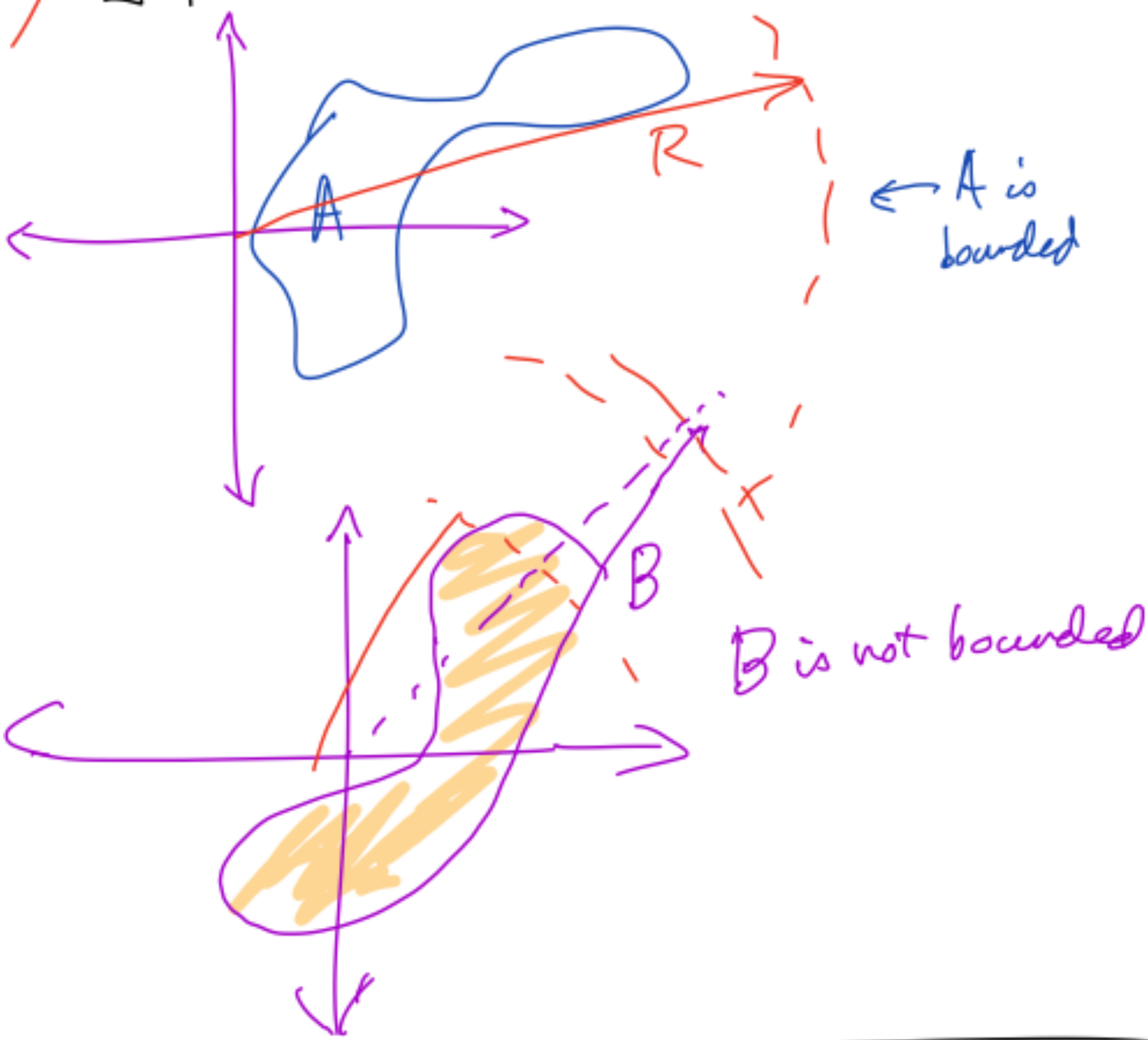
An <u>exterior point</u> of a set $S \subseteq \mathbb{C}$ is a point w of S^c s.t. $\exists \varepsilon > 0$ s.t. $B(w, \varepsilon) \subseteq S^c$. That is, an exterior point of S is an interior point of S^c.

A <u>Boundary point</u> of a set $S \subseteq \mathbb{C}$ is a point $z \in \mathbb{C}$ s.t. $\forall \varepsilon > 0$, $B(z, \varepsilon)$ contains at least one point of S and at least one point of S^c.

That is, a boundary point is neither an interior point nor an exterior point.

Both c & q are boundary points of S

A set $A \subseteq \mathbb{C}$ is called <u>Bounded</u> if $\exists\, R>0$ s.t. $A \subseteq B(0,R)$.

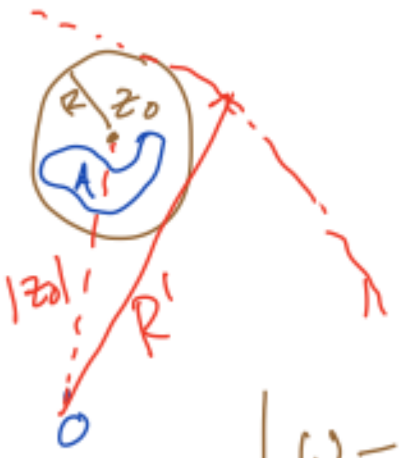


<u>Example</u>. Prove that $A \subseteq \mathbb{C}$ is bounded if and only if $\exists\, z_0 \in \mathbb{C}$, $R>0$ s.t. $A \subseteq B(z_0,R)$.

<u>Proof.</u> Suppose A is bounded. Then, by defn, $\exists\, R>0$ s.t. $A \subseteq B(0,R)$. Thus, the condition is satisfied with $z_0=0$.

Next, suppose that

$$\exists\, z_0 \in \mathbb{C},\ R > 0 \ \text{s.t.}\ A \subseteq B(z_0, R) \ \star$$



Then $\forall w \in A$,
$|w - z_0| < R$ $\Big\}\ \star\star$

But if we let
$R' = R + |z_0|$, then

$$|w - 0| = |w| = |w| - |z_0| + |z_0|$$

$$\underset{\boxed{\triangle\text{ inequality}}}{\leq} |w - z_0| + |z_0| < R + |z_0| = R'.$$

Thus, we have shown that

$\forall w \in A$, $|w| < R' \Rightarrow w \in B(0, R')$.

$\therefore A \subseteq B(0, R')$. Thus A is bounded. ▨

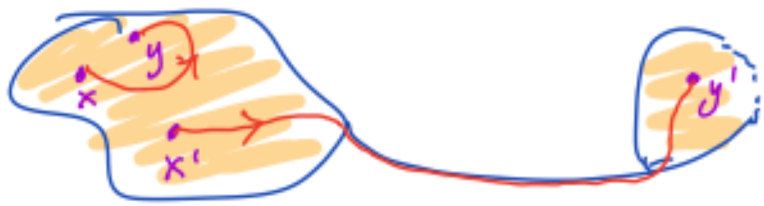
A connected set $C \subseteq \mathbb{C}$ is a set such that $\forall x, y \in C$, $\exists$ a path $\alpha : [0,1] \to C$ such that

$\alpha(0) = x$ and $\alpha(1) = y$.

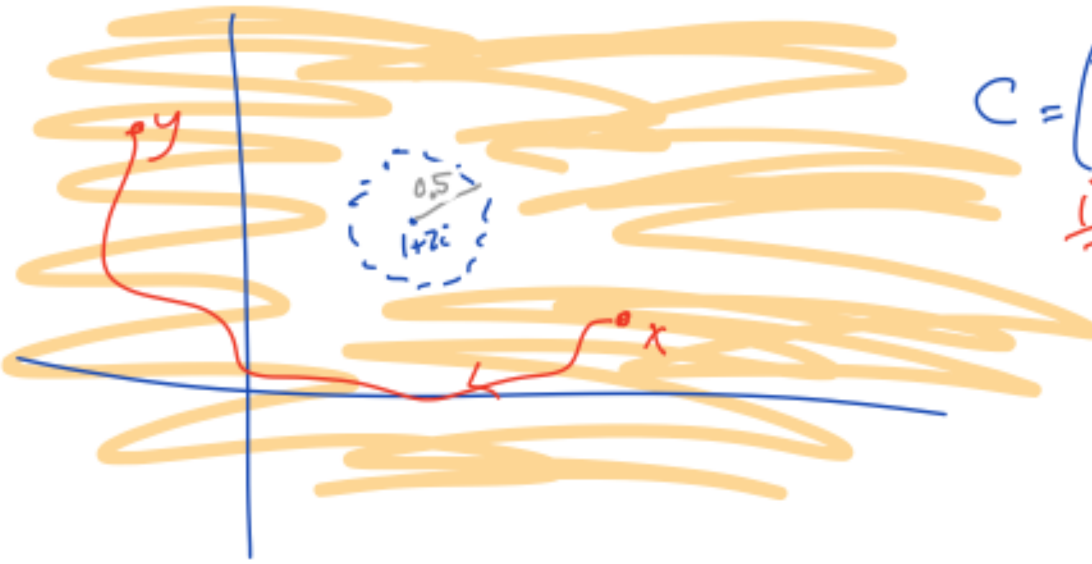
[path: continuous function from $[a,b]$ to $\mathbb{C}$.]
any interval

A is not connected

no path connecting x' & y'.



B is connected

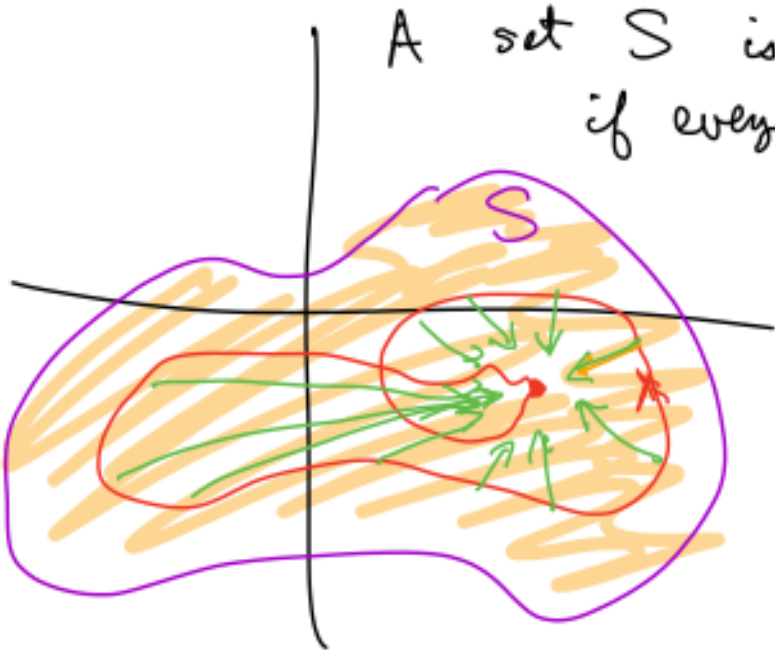


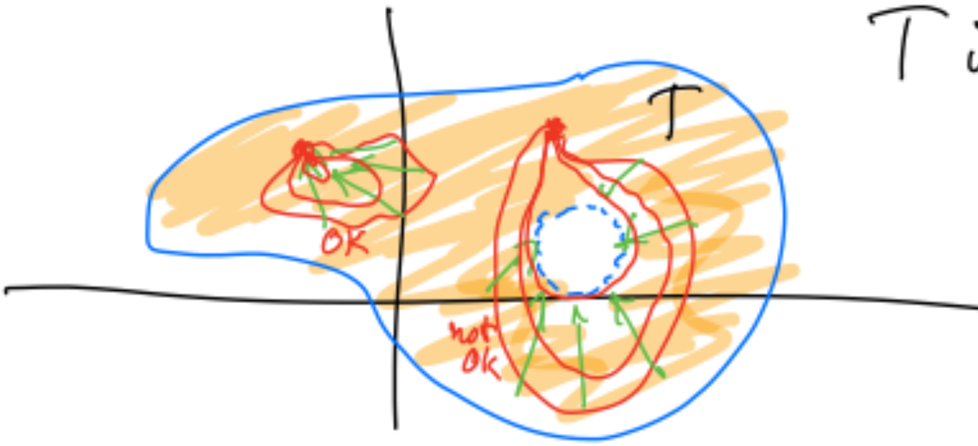
$C = \left(B(1+2i, 0.5)\right)^c$

is connected

A set S is simply connected if every closed path is contractible to a point in S.

within S.





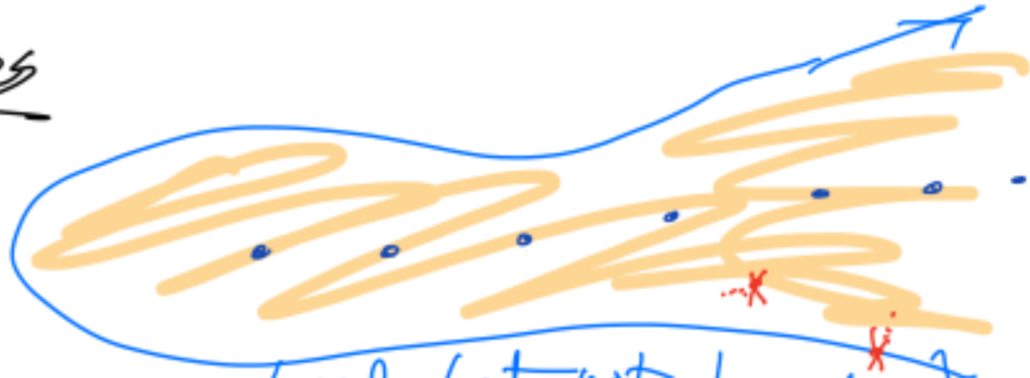
T is not simply connected

A set $F \subseteq \mathbb{C}$ is called <u>compact</u> if every open cover of F by open sets has a finite subcover.

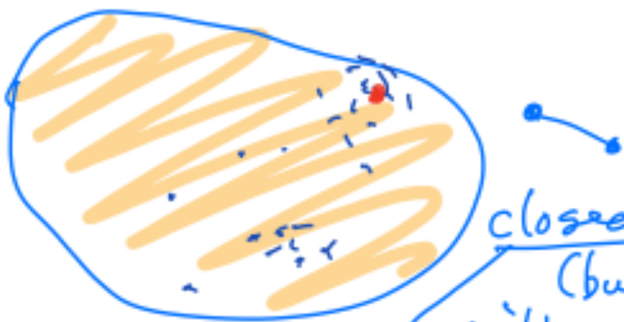
<u>Heine-Borel Theorem</u>:- A set $F \subseteq \mathbb{C}$ is compact if and only if it is both closed and bounded.

<u>Prop</u>:- A set $F \subseteq \mathbb{C}$ is compact if and only if every sequence $(z_n)_{n \in \mathbb{N}}$ of points in F has a subsequence that converges to a point in F.

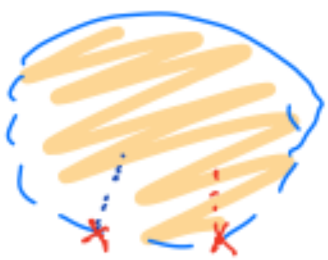
Examples



closed, but not bounded
not compact



closed and bounded
(but not connected)
→ it's compact.



bounded, not closed
not compact.

Bolzano-Weierstrass Thm.

Every bounded sequence in $\mathbb{C}$ has a subsequence that converges to a point in $\mathbb{C}$.

Nested Compact Sets Property.

If $C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$ is a nested sequence of compact sets, then

$$\bigcap_{j=1}^{\infty} C_j \text{ is nonempty}$$

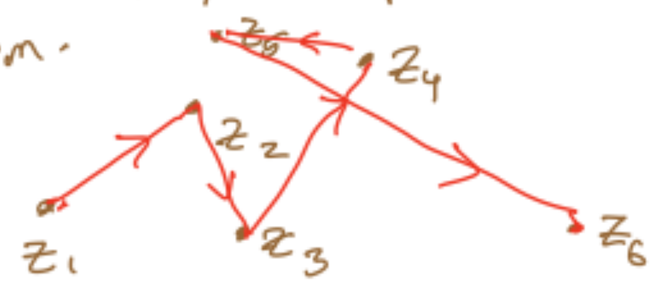
and is compact.

A *line segment* is a path that can be parametrized for $z_1 \neq z_2 \in \mathbb{C}$

by $\alpha(t) = z_1(1-t) + tz_2$.

$\alpha : [0,1] \to \mathbb{C}$. $\alpha(0) = z_1$, $\alpha(1) = z_2$.

A piecewise polygonal path is one of the form.



Each part of the path is a line segment.

Closed path = loop

path such that starting pt = ending pt.